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SHIFT-INVARIANT SPACES AND LINEAR OPERATOR EQUATIONS

BY

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ABSTRACT

In this paper we investigate the structure of finitely generated shift-invariant spaces and solvability of linear operator equations. Fourier transforms and semi-convolutions are used to characterize shift-invariant spaces. Criteria are provided for solvability of linear operator equations, including linear partial difference equations and discrete convolution equations. The results are then applied to the study of local shift-invariant spaces. Moreover, the approximation order of a local shift-invariant space is characterized under some mild conditions on the generators.

1. Introduction

The purpose of this paper is to investigate the structure of finitely generated shift-invariant spaces and solvability of linear operator equations. Our emphasis will be placed on finitely generated local shift-invariant spaces, that is, shift-invariant spaces generated by a finite number of compactly supported functions. It will be demonstrated that linear operator equations play an important role in the study of local shift-invariant spaces. Fourier transforms and semi-convolutions will be used to characterize shift-invariant spaces. Moreover, the approximation order of a local shift-invariant space will be characterized under some mild conditions on the generators.

A linear space S of functions from $\mathbb{R}^s$ to $\mathbb{C}$ is called **shift-invariant** if it is invariant under shifts (multi-integer translates), that is,

$$f \in S \quad \Longrightarrow \quad f(\cdot - \alpha) \in S \qquad \forall \alpha \in \mathbb{Z}^s.$$

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Let Φ be a set of functions from $\mathbb{R}^s$ to $\mathbb{C}$. We denote by $S_0(\Phi)$ the linear span of the shifts of the functions in Φ. Then $S_0(\Phi)$ is the smallest shift-invariant space containing Φ.

Let f be a (Lebesgue) measurable function on $\mathbb{R}^s$. For $1 \le p < \infty$, let

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, let $\|f\|_\infty$ be the essential supremum of f on $\mathbb{R}^s$. We denote by $L_p(\mathbb{R}^s)$ the Banach space of all measurable functions f on $\mathbb{R}^s$ such that $\|f\|_p$ is finite.

If Φ is a subset of $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$, we write $S_p(\Phi)$ for the closure of $S_0(\Phi)$ in $L_p(\mathbb{R}^s)$. Thus, $S_p(\Phi)$ is the smallest *closed* shift-invariant subspace of $L_p(\mathbb{R}^s)$ that contains Φ. The functions in Φ are called the **generators** of $S_p(\Phi)$. If Φ is a *finite* subset of $L_p(\mathbb{R}^s)$, then $S_p(\Phi)$ is said to be a **finitely generated shift-invariant space**. In particular, if Φ consists of a single function ϕ, then $S_p(\phi)$ is called a **principal shift-invariant space** (see [3]).

There are two ways to describe the structure of a finitely generated shift-invariant space. One way is to use the Fourier transforms of the generators. The other way is to employ the semi-convolutions of the generators with sequences on $\mathbb{Z}^s$.

If $f \in L_1(\mathbb{R}^s)$, the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi} \, dx, \qquad \xi \in \mathbb{R}^s.$$

The domain of the Fourier transform can be naturally extended to include $L_2(\mathbb{R}^s)$.

In [4], de Boor, DeVore and Ron gave the following characterization of a finitely generated shift-invariant subspace of $L_2(\mathbb{R}^s)$ in terms of the Fourier transforms of the generators. For a finite subset Φ of $L_2(\mathbb{R}^s)$ and a function $f \in L_2(\mathbb{R}^s)$, f lies in $S_2(\Phi)$ if and only if

$$\hat{f} = \sum_{\phi\in\Phi} \tau_\phi \hat{\phi}$$

for some 2π-periodic functions τ_ϕ, $\phi \in \Phi$.

The proof of this result given in [4] relies on the known characterization of doubly-invariant spaces (see [8]). On the other hand, the special case of a principal shift-invariant space was treated in [3] without recourse to the general theory of doubly-invariant spaces developed in [8]. In Section 2 we will give a simple proof of this result without appeal to the general tools used in [8] such as the range function and the pointwise projection.

It is also interesting to use semi-convolution to describe the structure of a finitely generated shift-invariant space. A function from $\mathbb{Z}^s$ to $\mathbb{C}$ is called a **sequence**. Let $\ell(\mathbb{Z}^s)$ denote the linear space of all sequences on $\mathbb{Z}^s$, and let $\ell_0(\mathbb{Z}^s)$ denote the linear space of all *finitely supported* sequences on $\mathbb{Z}^s$. Given a function $\phi\colon \mathbb{R}^s \to \mathbb{C}$ and a sequence $a \in \ell(\mathbb{Z}^s)$, the **semi-convolution** $\phi *' a$ is the sum

$$\sum_{\alpha\in\mathbb{Z}^s} \phi(\cdot - \alpha)a(\alpha).$$

This sum makes sense if either ϕ is compactly supported or a is finitely supported. Let Φ be a finite collection of compactly supported functions from $\mathbb{R}^s$ to $\mathbb{C}$. We use $S(\Phi)$ to denote the linear space of functions of the form $\sum_{\phi\in\Phi} \phi*' a_\phi$, where a_ϕ ($\phi \in \Phi$) are sequences on $\mathbb{Z}^s$. Following [4] we say that $S(\Phi)$ is **local**. For local shift-invariant spaces, see the work of Dahmen and Micchelli [5], de Boor, DeVore and Ron [4] and Jia [9].

Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$). In Section 3 we will prove that $S(\Phi) \cap L_p(\mathbb{R}^s)$ is always closed in $L_p(\mathbb{R}^s)$; hence $S_p(\Phi)$ is a subspace of $S(\Phi) \cap L_p(\mathbb{R}^s)$. In Section 4 we will show that $S(\Phi) \cap L_2(\mathbb{R}^s) = S_2(\Phi)$. Consequently, a function $f \in L_2(\mathbb{R}^s)$ lies in $S_2(\Phi)$ if and only if

$$f = \sum_{\phi\in\Phi} \phi*' a_\phi$$

for some sequences a_ϕ on $\mathbb{Z}^s$, $\phi \in \Phi$. For general p, it is a difficult question whether the two spaces $S(\Phi) \cap L_p(\mathbb{R}^s)$ and $S_p(\Phi)$ are the same.

When $s = 1$, it was proved in [11] that $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R})$ for $1 < p < \infty$. The essence of the proof given in [11] rests on the fact that $S(\Phi)$ has linearly independent generators. In Section 7 we extend this result to the case where Φ consists of a finite number of compactly supported functions in $L_p(\mathbb{R}^s)$ whose shifts are stable. Under such a condition we will show that $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R}^s)$ for $1 \le p < \infty$. When $p = \infty$, a modified result is also valid.

The results of Section 7 are based on a study of discrete convolution equations. As a matter of fact, discrete convolution equations can be viewed as linear partial difference equations with constant coefficients. In turn, linear partial difference and differential equations are special forms of linear operator equations. Section 5 is devoted to an investigation of linear operator equations. The general setting is as follows. Let V be a linear space over a field K, and let Λ be a ring of commuting linear operators on V. Consider the following system of linear operator equation:

$$\sum_{k=1}^{n} \lambda_{jk} u_k = v_j, \qquad j = 1, \ldots, m,$$

where $\lambda_{jk} \in \Lambda$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, $v_1, \ldots, v_m \in V$, and $u_1, \ldots, u_n$ are the unknowns. We will give a criterion for solvability of such linear operator equations. The result is then applied to linear partial difference and differential equations. On the basis of Section 5, we will establish a criterion for solvability of discrete convolution equations in Section 6.

Finally, the study of linear operator equations will be used to investigate approximation by shift-invariant spaces. Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. Let r be a positive integer. If Φ consists of a single function ϕ with $\hat{\phi}(0) \neq 0$, then it is known that $S(\phi)$ provides approximation order r if and only if $S(\phi)$ contains all polynomials of degree less than r. This result was established by Ron [17] for the case $p = \infty$, and by Jia [9] for the general case $1 \le p \le \infty$. In Section 8 we extend their results to finitely generated shift-invariant spaces. Let $\Phi = \{\phi_1, \ldots, \phi_n\}$. Suppose the sequences $(\hat{\phi}_k(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $k = 1, \ldots, n$, are linearly independent. Under this condition we will prove in Section 8 that $S(\Phi)$ provides approximation order r if and only if $S(\Phi)$ contains all polynomials of degree less than r.

2. Shift-invariant subspaces of $L_2(\mathbb{R}^s)$

In this section we give a new proof for the following result established by de Boor, DeVore and Ron in [4].

THEOREM 2.1: *Let Φ be a finite subset of $L_2(\mathbb{R}^s)$ and $f \in L_2(\mathbb{R}^s)$. Then $f \in S_2(\Phi)$ if and only if*

$$\hat{f} = \sum_{\phi\in\Phi} \tau_\phi \hat{\phi}$$

for some 2π-periodic functions τ_ϕ, $\phi \in \Phi$.

Recall that $L_2(\mathbb{R}^s)$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle := \int_{\mathbb{R}^s} f(x)\overline{g(x)}dx, \qquad f, g \in L_2(\mathbb{R}^s),$$

where $\overline{g}$ denotes the complex conjugate of g. We say that f is **orthogonal** to g if $\langle f, g\rangle = 0$. The orthogonal complement of a subspace V of a Hilbert space is denoted by $V^\perp$. It is easily seen that $S_2(f)$ is orthogonal to $S_2(g)$ if and only if $\langle f(\cdot - \alpha), g\rangle = 0$ for all $\alpha \in \mathbb{Z}^s$. The following lemma follows from basic properties of the Fourier transform.

LEMMA 2.2: *If $f, g \in L_2(\mathbb{R}^s)$, then the series*

$$h(\xi) := \sum_{\beta\in\mathbb{Z}^s} \hat{f}(\xi + 2\pi\beta)\overline{\hat{g}(\xi + 2\pi\beta)}$$

converges absolutely for almost every $\xi \in \mathbb{R}^s$. *The Fourier coefficients of the* 2π*-periodic function* h *are* $\langle f(\cdot - \alpha), g\rangle$, $\alpha \in \mathbb{Z}^s$.

The **bracket product** of two functions f and g in $L_2(\mathbb{R}^s)$ is defined by

$$[f,g](e^{i\xi}) := \sum_{\beta\in\mathbb{Z}^s} \hat{f}(\xi + 2\pi\beta)\overline{\hat{g}(\xi + 2\pi\beta)}, \qquad \xi \in \mathbb{R}^s.$$

In particular, if f and g are compactly supported, then

$$(2.1) \qquad [f,g](e^{i\xi}) = \sum_{\alpha\in\mathbb{Z}^s} \langle f(\cdot - \alpha), g\rangle e^{i\alpha\cdot\xi} \qquad \forall\xi \in \mathbb{R}^s.$$

The bracket product $[f,g]$ was introduced in [14] (see Theorem 3.2 there) under some mild decay conditions on f and g. This restriction was removed in [3].

The bracket product turns out to be a convenient tool in the study of orthogonality in $L_2(\mathbb{R}^s)$. The following lemma is an easy consequence of Lemma 2.2.

LEMMA 2.3: *If* $\phi, \psi \in L_2(\mathbb{R}^s)$, *then* ψ *is orthogonal to* $S_2(\phi)$ *if and only if* $[\phi,\psi](e^{i\xi}) = 0$ *for almost every* $\xi \in \mathbb{R}^s$. *Moreover,* $\{\phi(\cdot - \alpha)\colon \alpha \in \mathbb{R}^s\}$ *forms an orthonormal system if and only if* $[\phi,\phi](e^{i\xi}) = 1$ *for almost every* $\xi \in \mathbb{R}^s$.

Proof of Theorem 2.1: Denote by $F(\Phi)$ the linear space of those functions $f \in L_2(\mathbb{R}^s)$ for which $\hat{f} = \sum_{\phi\in\Phi} \tau_\phi\hat{\phi}$ for some 2π-periodic functions τ_ϕ ($\phi \in \Phi$). Suppose $f \in F(\Phi)$. Then for every $g \in S_2(\Phi)^\perp$ and almost every $\xi \in \mathbb{R}^s$,

$$[f,g](e^{i\xi}) = \sum_{\phi\in\Phi} \tau_\phi(\xi)[\phi,g](e^{i\xi}) = 0.$$

By Lemma 2.3, this shows that f is orthogonal to $S_2(\Phi)^\perp$; hence $f \in S_2(\Phi)^{\perp\perp} = S_2(\Phi)$. In other words, $F(\Phi) \subseteq S_2(\Phi)$.

For the proof of $F(\Phi) \supseteq S_2(\Phi)$ we shall proceed by induction on $\#\Phi$, the number of elements in Φ. Suppose $\#\Phi = 1$ and $\Phi = \{\phi\}$. In order to prove $F(\phi) = S_2(\phi)$, it suffices to show that $F(\phi)$ is closed. Suppose f lies in the closure of $F(\phi)$. Then there exists a sequence $(f_n)_{n=1,2,\ldots}$ in $F(\phi)$ such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. It follows that $\|\hat{f}_n - \hat{f}\|_2 \to 0$. By passing to a subsequence if necessary, we may assume that $\hat{f}_n$ converges to $\hat{f}$ almost everywhere. Since $f_n \in F(\phi)$, $\hat{f}_n = \tau_n\hat{\phi}$ for some 2π-periodic function τ_n. Let

$$E := \{\xi \in \mathbb{R}^s\colon \hat{\phi}(\xi + 2\beta\pi) = 0 \quad \forall\beta \in \mathbb{Z}^s\}.$$

Then E is 2π-periodic, i.e., $\xi \in E$ implies $\xi + 2\beta\pi \in E$ for all $\beta \in \mathbb{Z}^s$. For each n, let

$$\lambda_n(\xi) := \begin{cases} \tau_n(\xi) & \text{if } \xi \notin E, \\ 0 & \text{if } \xi \in E. \end{cases}$$

Evidently, λ_n is 2π-periodic and $\hat{f}_n = \lambda_n \hat{\phi}$. Since $\hat{f}_n$ converges to $\hat{f}$ almost everywhere, for almost every $\xi \in \mathbb{R}^s$, $\lim_{n\to\infty} \hat{f}_n(\xi + 2\beta\pi) = \hat{f}(\xi + 2\beta\pi)$ for all $\beta \in \mathbb{Z}^s$. Let ξ be such a point. If $\xi \in E$, then $\lambda_n(\xi) = 0$ for all n. If $\xi \notin E$, then $\hat{\phi}(\xi + 2\beta\pi) \neq 0$ for some $\beta \in \mathbb{Z}^s$. Consequently,

$$\lim_{n\to\infty} \lambda_n(\xi) = \lim_{n\to\infty} \hat{f}_n(\xi + 2\beta\pi)/\hat{\phi}(\xi + 2\beta\pi) = \hat{f}(\xi + 2\beta\pi)/\hat{\phi}(\xi + 2\beta\pi).$$

This shows that $\lambda(\xi) := \lim_{n\to\infty} \lambda_n(\xi)$ exists for almost every $\xi \in \mathbb{R}^s$. Since each λ_n is 2π-periodic, the limit function λ is also 2π-periodic. Taking limits of both sides of $\hat{f}_n = \lambda_n \hat{\phi}$, we obtain $\hat{f} = \lambda\hat{\phi}$. This shows that $f \in F(\phi)$. Therefore $F(\phi)$ is closed.

Now assume that $F(\Phi) = S_2(\Phi)$ and we wish to prove that $F(\Phi \cup \psi) \supseteq S_2(\Phi \cup \psi)$ for any $\psi \in L_2(\mathbb{R}^s)$. Let P_Φ denote the orthogonal projection of $L_2(\mathbb{R}^s)$ onto $S_2(\Phi)$, and let 1 denote the identity operator on $L_2(\mathbb{R}^s)$. Let $\rho := (1 - P_\Phi)\psi$. Then $S_2(\Phi)$ is orthogonal to $S_2(\rho)$, and hence $S_2(\Phi) + S_2(\rho)$ is closed. With $g := P_\Phi \psi \in S_2(\Phi) = F(\Phi)$ we have $\rho = \psi - g \in F(\Phi \cup \psi)$, and so $S_2(\rho) = F(\rho) \subseteq F(\Phi \cup \psi)$. But $S_2(\Phi) + S_2(\rho)$ is closed, and $\psi = g + \rho \in S_2(\Phi) + S_2(\rho)$. Therefore we have

$$F(\Phi \cup \psi) \supseteq S_2(\Phi) + S_2(\rho) \supseteq S_2(\Phi \cup \psi).$$

This completes the induction procedure. ∎

3. Local shift-invariant spaces

Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ ($1 \leq p \leq \infty$). In this section we shall show that $S(\Phi) \cap L_p(\mathbb{R}^s)$ is closed in $L_p(\mathbb{R}^s)$.

A measurable function $f\colon \mathbb{R}^s \to \mathbb{C}$ is called **locally integrable** if

$$\|f\|_1(K) := \int_K |f(x)|dx < \infty$$

for every compact subset K of $\mathbb{R}^s$. We denote by $L_{loc} := L_{loc}(\mathbb{R}^s)$ the linear space of all locally integrable functions on $\mathbb{R}^s$. For $k = 1, 2, \ldots$, the functional p_k given by

$$p_k(f) := \int_{[-k,k]^s} |f(x)|dx$$

is a semi-norm on L_{loc}. The family of semi-norms $\{p_k: k = 1, 2, \ldots\}$ induces a topology on L_{loc} so that L_{loc} becomes a complete, metrizable, locally convex topological vector space. In other words, L_{loc} is a Fréchet space (see, e.g., [7, p. 160]). Let $(f_n)_{n=1,2,\ldots}$ be a sequence in L_{loc}. Then f_n converges to a function $f \in L_{loc}$ if and only if for every compact subset K of $\mathbb{R}^s$, $\|f_n - f\|_1(K) \to 0$ as $n \to \infty$.

The following theorem shows that a finitely generated local shift-invariant subspace of $L_{loc}(\mathbb{R}^s)$ is closed in it.

THEOREM 3.1: *Let Φ be a finite collection of compactly supported integrable functions on $\mathbb{R}^s$. Then $S(\Phi)$ is a closed subspace of $L_{loc}(\mathbb{R}^s)$. If Φ is a subset of $L_p(\mathbb{R}^s)$ for some p, $1 \le p \le \infty$, then $S(\Phi) \cap L_p(\mathbb{R}^s)$ is closed in $L_p(\mathbb{R}^s)$.*

The proof of Theorem 3.1 is based on the author's recent paper [9] on approximation order of shift-invariant spaces. Let us recall some results from [9].

Let $S = S(\Phi)$, where Φ is a finite collection of compactly supported integrable functions on $\mathbb{R}^s$. The restriction of S to the cube $[0,1)^s$ is finite dimensional. Thus, we can find a finite collection $\{\psi_i: i \in I\}$ of integrable functions on $\mathbb{R}^s$ such that each ψ_i vanishes outside $[0,1)^s$ and $\{\psi_i|_{[0,1)^s}: i \in I\}$ forms a basis for $S|_{[0,1)^s}$. The shifts of the functions ψ_i $(i \in I)$ are linearly independent; that is, for sequences $a_i \in \ell(\mathbb{Z}^s)$ $(i \in I)$,

$$\sum_{i \in I} \psi_i *' a_i = 0 \quad \Longrightarrow \quad a_i = 0 \qquad \forall i \in I.$$

Let $\ell(\mathbb{Z}^s)^I$ denote the linear space of all mappings from I to $\ell(\mathbb{Z}^s)$. We define the linear mapping T from $\ell(\mathbb{Z}^s)^I$ to $L_{loc}(\mathbb{R}^s)$ as follows:

$$T(a) := \sum_{i \in I} \psi_i *' a_i \qquad \text{for } a = (a_i)_{i \in I} \in \ell(\mathbb{Z}^s)^I.$$

Let V be the range of the mapping T. Then $S(\Phi)$ is a linear subspace of V.

A function $f \in V$ has the following representation:

$$f = \sum_{i \in I} \psi_i *' f_i, \tag{3.1}$$

where $f_i \in \ell(\mathbb{Z}^s)$, $i \in I$. Suppose $\Phi = \{\phi_j: j \in J\}$. Then f lies in $S(\Phi)$ if and only if there exist sequences u_j $(j \in J)$ on $\mathbb{Z}^s$ such that

$$f = \sum_{j \in J} \phi_j *' u_j. \tag{3.2}$$

Since each ϕ_j is compactly supported and belongs to V, we can find finitely supported sequences c_{ij} on $\mathbb{Z}^s$ $(i \in I, j \in J)$ such that

$$\phi_j = \sum_{i\in I} \psi_i *' c_{ij}, \qquad j \in J. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} f &= \sum_{j\in J}\sum_{\alpha\in\mathbb{Z}^s} \phi_j(\cdot - \alpha)u_j(\alpha) \\ &= \sum_{j\in J}\sum_{\alpha\in\mathbb{Z}^s}\sum_{i\in I}\sum_{\beta\in\mathbb{Z}^s} \psi_i(\cdot - \alpha - \beta)c_{ij}(\beta)u_j(\alpha) \\ &= \sum_{i\in I}\sum_{\alpha\in\mathbb{Z}^s}\left[\sum_{j\in J}\sum_{\beta\in\mathbb{Z}^s} c_{ij}(\beta)u_j(\alpha-\beta)\right]\psi_i(\cdot - \alpha). \end{aligned}$$

Comparing this with (3.1), we conclude that f lies in $S(\Phi)$ if and only if

$$\sum_{j\in J}\sum_{\beta\in\mathbb{Z}^s} c_{ij}(\beta)u_j(\alpha-\beta) = f_i(\alpha) \qquad \forall\alpha\in\mathbb{Z}^s \text{ and } i \in I. \tag{3.4}$$

Given $\alpha \in \mathbb{Z}^s$, we denote by τ^α the difference operator on $\ell(\mathbb{Z}^s)$ given by

$$\tau^\alpha a := a(\cdot + \alpha), \qquad a \in \ell(\mathbb{Z}^s).$$

If $p = \sum_{\alpha\in\mathbb{Z}^s} c_\alpha z^\alpha$ is a Laurent polynomial, where $c_\alpha = 0$ except for finitely many α, then p induces the difference operator

$$p(\tau) := \sum_{\alpha\in\mathbb{Z}^s} c_\alpha \tau^\alpha.$$

For $i \in I$ and $j \in J$, let g_{ij} denote the Laurent polynomial given by

$$g_{ij}(z) = \sum_{\beta\in\mathbb{Z}^s} c_{ij}(\beta) z^{-\beta}, \qquad z \in (\mathbb{C}\setminus\{0\})^s.$$

Then (3.4) can be rewritten as

$$\sum_{j\in J} g_{ij}(\tau)u_j = f_i, \qquad i \in I. \tag{3.5}$$

We observe that (3.5) is a system of linear partial difference equations with constant coefficients. This system of partial difference equations is said to be **consistent** if it can be solved for $(u_j)_{j\in J}$. It is said to be **compatible** if the following compatibility conditions are satisfied: For Laurent polynomials q_i ($i \in I$),

$$\sum_{i\in I} q_i g_{ij} = 0 \quad \forall j \in J \quad \Longrightarrow \quad \sum_{i\in I} q_i(\tau) f_i = 0.$$

It was proved in [9, Theorem 3] that the system of partial difference equations in (3.5) is consistent if and only if it is compatible. This result is an extension of the well-known Ehrenpreis principle for solvability of linear partial differential equations with constant coefficients (see [6]).

Now let P denote the set of all Laurent polynomials in s variables. Let Q be the subset of P^I given by

$$Q := \left\{ (q_i)_{i\in I} \colon \sum_{i\in I} q_i g_{ij} = 0 \quad \forall j \in J \right\}.$$

Thus, we arrive at the following conclusion.

LEMMA 3.2: *There exists a subset Q of P^I such that a function $f = \sum_{i\in I} \psi_i *' f_i$ lies in $S(\Phi)$ if and only if*

$$\sum_{i\in I} q_i(\tau) f_i = 0 \qquad \forall (q_i)_{i\in I} \in Q. \tag{3.6}$$

We are in a position to establish Theorem 3.1.

Proof of Theorem 3.1: First, we show that V is a closed linear subspace of $L_{loc}(\mathbb{R}^s)$. Since $\{\psi_i|_{[0,1)^s} \colon i \in I\}$ is linearly independent, there exist two positive constants C_1 and C_2 such that for $f = T(a) \in V$ and for all $\beta \in \mathbb{Z}^s$,

$$C_1 \sum_{i\in I} |a_i(\beta)| \le \|f\|_1(\beta + [0,1)^s) \le C_2 \sum_{i\in I} |a_i(\beta)|. \tag{3.7}$$

Let $(f^{(n)})_{n=1,2,\ldots}$ be a sequence in V converging to a function f in $L_{loc}(\mathbb{R}^s)$. Suppose $f^{(n)} = T(a^{(n)})$. Then by (3.7) we have

$$\left|a_i^{(m)}(\beta) - a_i^{(n)}(\beta)\right| \le C_1^{-1} \|f^{(m)} - f^{(n)}\|_1(\beta + [0,1)^s)$$

for all $i \in I$ and $\beta \in \mathbb{Z}^s$. This shows that $(a_i^{(n)}(\beta))_{n=1,2,\ldots}$ is a Cauchy sequence of complex numbers. Let $a_i(\beta) := \lim_{n\to\infty} a_i^{(n)}(\beta)$ and $a := (a_i)_{i\in I}$. Using (3.7) again, we see that for all $\beta \in \mathbb{Z}^s$

$$\|T(a) - T(a^{(n)})\|_1(\beta + [0,1)^s) \le C_2 \sum_{i\in I} \left|a_i(\beta) - a_i^{(n)}(\beta)\right|.$$

Hence $T(a^{(n)})$ converges to $T(a)$ in $L_{loc}(\mathbb{R}^s)$. In other words, $f = T(a)$, thereby proving that V is closed in $L_{loc}(\mathbb{R}^s)$.

Next, we show that $S(\Phi)$ is closed in V. Let $\left(f^{(n)}\right)_{n=1,2,\dots}$ be a sequence in $S(\Phi)$ converging to $f \in V$. Suppose $f^{(n)} = T(a^{(n)})$ for each n and $f = T(a)$. Then the preceding paragraph tells us that for each $i \in I$ and each $\beta \in \mathbb{Z}^s$, $a_i^{(n)}(\beta)$ converges to $a_i(\beta)$ as $n \to \infty$. In other words, $a_i^{(n)}$ converges to a_i pointwise. Since each $f^{(n)}$ lies in $S(\Phi)$, by Lemma 3.2 we have

$$\sum_{i\in I} q_i(\tau)a_i^{(n)} = 0 \qquad \forall (q_i)_{i\in I} \in Q. \tag{3.8}$$

For a fixed element $(q_i)_{i\in I} \in Q$ and a fixed $\beta \in \mathbb{Z}^s$, $q_i(\tau)a_i(\beta)$ only involves finitely many $a_i(\alpha)$, $\alpha \in \mathbb{Z}^s$. Letting $n \to \infty$ in (3.8) we conclude that

$$\sum_{i\in I} q_i(\tau)a_i = 0 \qquad \forall (q_i)_{i\in I} \in Q.$$

This shows that $f = T(a)$ lies in $S(\Phi)$, by Lemma 3.2. Therefore, $S(\Phi)$ is a closed subspace of $L_{loc}(\mathbb{R}^s)$.

Finally, suppose that Φ is a subset of $L_p(\mathbb{R}^s)$ for some p, $1 \le p \le \infty$. If $(f^{(n)})_{n=1,2,\dots}$ is a sequence in $S(\Phi)\cap L_p(\mathbb{R}^s)$ converging to f in $L_p(\mathbb{R}^s)$, then $f^{(n)}$ converges to f in the topology of $L_{loc}(\mathbb{R}^s)$. Hence f lies in $S(\Phi)$ by what has been proved. This shows that $S(\Phi) \cap L_p(\mathbb{R}^s)$ is closed in $L_p(\mathbb{R}^s)$. ∎

4. Local shift-invariant subspaces of $L_2(\mathbb{R}^s)$

Let Φ be a finite collection of compactly supported functions in $L_2(\mathbb{R}^s)$. In [3] de Boor, DeVore and Ron demonstrated that the two spaces $S(\Phi) \cap L_2(\mathbb{R}^s)$ and $S_2(\Phi)$ provide the same approximation order. However, they left the question open whether these two spaces are the same. In this section we show that these two spaces are indeed the same. Consequently, we give a characterization for $S_2(\Phi)$ in terms of the semi-convolutions of the generators with sequences on $\mathbb{Z}^s$.

THEOREM 4.1: *Let Φ be a finite collection of compactly supported functions in $L_2(\mathbb{R}^s)$. Then $S(\Phi) \cap L_2(\mathbb{R}^s) = S_2(\Phi)$. Consequently, a function $f \in L_2(\mathbb{R}^s)$ lies in $S_2(\Phi)$ if and only if*

$$f = \sum_{\phi\in\Phi} \phi *' a_\phi$$

for some sequences a_ϕ on $\mathbb{Z}^s$, $\phi \in \Phi$.

In our proof we use the following two basic facts. First, if $f \in L_2(\mathbb{R}^s)$ and $a \in \ell_0(\mathbb{Z}^s)$, then

$$\widehat{f *' a}(\xi) = \hat{f}(\xi)\tilde{a}(e^{-i\xi}), \qquad \xi \in \mathbb{R}^s, \tag{4.1}$$

where $\tilde{a}(z) := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)z^\alpha$ is the **symbol** of a. Second, if $f \in L_2(\mathbb{R}^s)$ and $g = f*'a$ for some nontrivial sequence $a \in \ell_0(\mathbb{Z}^s)$, then $S_2(f) = S_2(g)$ (see [4, Corollary 2.5]). Indeed, $g = f*'a$ implies

$$\hat{g}(\xi) = \hat{f}(\xi)\tilde{a}(e^{-i\xi}) \quad \text{and} \quad \hat{f}(\xi) = \hat{g}(\xi)/\tilde{a}(e^{-i\xi}), \qquad \text{for a.e. } \xi \in \mathbb{R}^s,$$

where $\tilde{a}(e^{-i\xi})$ is a 2π-periodic trigonometric polynomial. So $f \in S_2(g)$ and $g \in S_2(f)$ by Theorem 2.1.

We also need the following lemma (cf. [14, Theorem 4.4] and [4, Theorem 3.38]).

LEMMA 4.2: *Let Φ be a finite collection of compactly supported functions in $L_2(\mathbb{R}^s)$, and let P_Φ denote the orthogonal projection of $L_2(\mathbb{R}^s)$ onto $S_2(\Phi)$. Then there exists a nontrivial sequence $b \in \ell_0(\mathbb{Z}^s)$ such that for every compactly supported function $g \in L_2(\mathbb{R}^s)$, $P_\Phi(g*'b)$ is compactly supported.*

Proof: The proof proceeds by induction on $\#\Phi$. Suppose Φ consists of a single function $\phi \neq 0$. For a compactly supported function $g \in L_2(\mathbb{R}^s)$, let h and u be the functions determined by

$$\hat{h}(\xi) = [\phi,\phi](e^{i\xi})\hat{g}(\xi) \quad \text{and} \quad \hat{u}(\xi) = [g,\phi](e^{i\xi})\hat{\phi}(\xi), \qquad \xi \in \mathbb{R}^s.$$

Let b and c be the sequences such that $\tilde{b}(e^{-i\xi}) = [\phi,\phi](e^{i\xi})$ and $\tilde{c}(e^{-i\xi}) = [g,\phi](e^{i\xi})$, $\xi \in \mathbb{R}^s$. Note that the sequence b is independent of g. Since both g and ϕ are compactly supported, (2.1) tells us that both b and c are finitely supported. Moreover, by (4.1) we have $h = g*'b$ and $u = \phi*'c$. We find that

$$[h-u,\phi] = [h,\phi] - [u,\phi] = [\phi,\phi][g,\phi] - [g,\phi][\phi,\phi] = 0,$$

so $u = P_\phi h = P_\phi(g*'b)$. But $u = \phi*'c$ is compactly supported.

Now assume that the lemma is valid for a finite set Φ of compactly supported functions in $L_2(\mathbb{R}^s)$. We wish to prove that it is also true for $\Phi \cup \psi$, where ψ is a compactly supported function in $L_2(\mathbb{R}^s)$. By the induction hypothesis, there exists a nontrivial sequence $b \in \ell_0(\mathbb{Z}^s)$ such that for every compactly supported function $g \in L_2(\mathbb{R}^s)$, $P_\Phi(g*'b)$ is compactly supported. Let $\rho := \psi*'b - P_\Phi(\psi*'b)$. Then ρ is compactly supported. Moreover, since $S_2(\psi) = S_2(\psi*'b)$, the space

$$S_2(\Phi\cup\psi) = S_2\big(\Phi\cup(\psi*'b)\big) = S_2(\Phi\cup\rho)$$

is the orthogonal sum of $S_2(\Phi)$ and $S_2(\rho)$. By what has been proved, there exists a nontrivial sequence c such that for every compactly supported function

$g \in L_2(\mathbb{R}^s)$, $P_\rho(g{*}'c)$ is compactly supported. Note that $g{*}'(b*c) = (g{*}'b){*}'c = (g{*}'c){*}'b$. Therefore, for every compactly supported function $g \in L_2(\mathbb{R}^s)$,

$$P_{\Phi\cup\psi}(g *' (b*c)) = P_\Phi(g *' (b*c)) + P_\rho(g *' (b*c))$$

is compactly supported. ∎

Proof of Theorem 4.1: Theorem 3.1 shows $S(\Phi) \cap L_2(\mathbb{R}^s) \supseteq S_2(\Phi)$, so we only have to show $S_2(\Phi) \supseteq S(\Phi) \cap L_2(\mathbb{R}^s)$. The latter was proved in [3, Theorem 2.16] for the case $\#\Phi = 1$. For the general case we argue as follows. Let

$$f = \sum_{\phi\in\Phi} \phi{*}'a_\phi \in S(\Phi) \cap L_2(\mathbb{R}^s).$$

We wish to prove $f \in S_2(\Phi)$. For this purpose, we observe that for every compactly supported function $g \in S_2(\Phi)^\perp$,

$$\langle f, g\rangle = \sum_{\phi\in\Phi}\sum_{\alpha\in\mathbb{Z}^s} \langle \phi(\cdot-\alpha), g\rangle a_\phi(\alpha) = 0.$$

By Lemma 4.2 we can find a nontrivial sequence $b \in \ell_0(\mathbb{Z}^s)$ such that for every function $h \in L_2(\mathbb{R}^s)$ with compact support, $P_\Phi(h{*}'b)$ is compactly supported. Let $g \in S_2(\Phi)^\perp$. Then $P_\Phi(g{*}'b) = 0$. There exists a sequence $(g_n)_{n=1,2,\ldots}$ of compactly supported functions in $L_2(\mathbb{R}^s)$ such that $\|g_n - g\|_2 \to 0$ as $n \to \infty$. Let

$$h_n := g_n{*}'b - P_\Phi(g_n{*}'b).$$

Then each h_n is compactly supported and $h_n \in S_2(\Phi)^\perp$. Hence $\langle f, h_n\rangle = 0$ for $n = 1, 2, \ldots$. Furthermore,

$$\lim_{n\to\infty} \langle f, h_n\rangle = \langle f, g{*}'b - P_\Phi(g{*}'b)\rangle = \langle f, g{*}'b\rangle.$$

This shows $\langle f, g{*}'b\rangle = 0$. Let c be the sequence given by $c(\alpha) = \overline{b(-\alpha)}$ for all $\alpha \in \mathbb{Z}^s$. Then $\langle f, g{*}'b\rangle = 0$ implies

$$\begin{aligned}\langle f{*}'c, g\rangle &= \sum_{\alpha\in\mathbb{Z}^s} \langle f(\cdot-\alpha), g\rangle c(\alpha)\\ &= \sum_{\alpha\in\mathbb{Z}^s} \langle f, g(\cdot+\alpha)\overline{c(\alpha)}\rangle = \langle f, g{*}'b\rangle = 0.\end{aligned}$$

This is true for all $g \in S_2(\Phi)^\perp$; hence $f{*}'c \in S_2(\Phi)^{\perp\perp} = S_2(\Phi)$. Therefore we have $f \in S_2(f{*}'c) \subseteq S_2(\Phi)$, as desired. ∎

5. Linear operator equations

In this section we establish a criterion for solvability of linear operator equations and then apply the result to linear partial difference and differential equations with constant coefficients. The study of linear operator equations is important for our investigation of local shift-invariant spaces.

Let K be a field, and let V be a linear space over K. Given a linear mapping λ on V, we use $\ker\lambda$ to denote its kernel $\{u \in V: \lambda u = 0\}$. Thus, λ is one-to-one if $\ker\lambda = \{0\}$. If λ is both one-to-one and onto, then we say that λ is **invertible**.

Let $L(V)$ be the set of all linear mappings on V. Then $L(V)$ is a ring under addition and composition. The identity mapping on V is the identity element of $L(V)$. In general, $L(V)$ is noncommutative.

We are interested in commutative subrings of $L(V)$ with identity. Let Λ be such a subring. The ideal generated by finitely many elements $\lambda_1, \dots, \lambda_m$ in Λ is denoted by $(\lambda_1, \dots, \lambda_m)$. An ideal I of Λ is said to be **invertible** if I contains an invertible linear mapping. Note that the inverse of an invertible linear mapping $\lambda \in \Lambda$ is not required to lie in Λ. The kernel of I, denoted $\ker I$, is the intersection of all $\ker\lambda$, $\lambda \in I$.

Consider the following system of linear operator equations:

$$\sum_{k=1}^{n} \lambda_{jk} u_k = v_j, \qquad j = 1, \dots, m, \tag{5.1}$$

where $\lambda_{jk} \in \Lambda$ for $j = 1, \dots, m$ and $k = 1, \dots, n$, $v_1, \dots, v_m \in V$, and $u_1, \dots, u_n$ are the unknowns. Our purpose is to give a criterion for solvability of (5.1). Linear operator equations with one unknown ($n = 1$) were investigated by Jia, Riemenschneider and Shen in [15].

We say that the system (5.1) is **consistent** if there exist $u_1, \dots, u_n \in V$ that satisfy the equations in (5.1). Two systems of linear operator equations are said to be **equivalent** if they have the same solutions. We say that (5.1) is **compatible** if for any $\mu_1, \dots, \mu_m \in \Lambda$ with $\sum_{j=1}^{m} \mu_j \lambda_{jk} = 0$, $k = 1, \dots, n$, one must have $\sum_{j=1}^{m} \mu_j v_j = 0$. Evidently, if (5.1) is consistent, then it is compatible.

If we replace every vector v_j ($j = 1, \dots, m$) in (5.1) by the zero vector, then the resulting system is called the **associated homogeneous system**. Thus, the solutions of (5.1) are unique if and only if the associated homogeneous system only has the trivial solution.

THEOREM 5.1: *Let Λ be a commutative subring of $L(V)$ with identity. Suppose that every finitely generated ideal I of Λ with $\ker I = \{0\}$ is invertible. Then the system (5.1) of linear operator equations is uniquely solvable for $u_1, \dots, u_n$ in V*

if and only if it is compatible and the associated homogeneous system only has the trivial solution.

Proof: It is obvious that the two conditions are necessary for the system (5.1) to be uniquely solvable. The proof of sufficiency proceeds by inductions on n.

Suppose $n = 1$ and consider the system of linear operator equations

$$\lambda_j u = v_j, \qquad j = 1, \ldots, m. \tag{5.2}$$

By the assumption, the associated homogeneous system

$$\lambda_j u = 0, \qquad j = 1, \ldots, m,$$

only has the trivial solution. In other words, $\ker(\lambda_1, \ldots, \lambda_m) = \{0\}$; hence $(\lambda_1, \ldots, \lambda_m)$ is invertible. Thus, there exist $\mu_1, \ldots, \mu_m \in \Lambda$ such that $\nu := \mu_1\lambda_1 + \cdots + \mu_m\lambda_m$ is invertible. Let

$$u := \nu^{-1}(\mu_1 v_1 + \cdots + \mu_m v_m).$$

We claim that u satisfies the equations in (5.2). Indeed, since (5.2) is compatible, we have $\lambda_j v_k = \lambda_k v_j$ for $j, k \in \{1, \ldots, m\}$. Therefore

$$\nu\lambda_j u = \lambda_j(\nu u) = \sum_{k=1}^{m} \lambda_j \mu_k v_k = \sum_{k=1}^{m} \mu_k(\lambda_j v_k) = \sum_{k=1}^{m} \mu_k \lambda_k v_j = \nu v_j.$$

But ν is invertible, so it follows that $\lambda_j u = v_j$ for $j = 1, \ldots, m$.

Let $n > 1$ and assume that the theorem has been verified for $n - 1$. We shall prove that (5.1) is uniquely solvable under the conditions stated in the theorem. Note that the kernel of the ideal $(\lambda_{11}, \ldots, \lambda_{m1})$ is trivial, for otherwise the associated homogeneous system would have nontrivial solutions. Thus, there exist $\mu_1, \ldots, \mu_m \in \Lambda$ such that the linear mapping $\nu := \mu_1\lambda_{11} + \cdots + \mu_m\lambda_{m1}$ is invertible. Apply ν to both sides of each equation in (5.1):

$$\sum_{k=1}^{n} \nu\lambda_{jk} u_k = \nu v_j, \qquad j = 1, \ldots, m. \tag{5.3}$$

Since ν is invertible, two systems (5.1) and (5.3) are equivalent. Let

$$v_0 := \sum_{j=1}^{m} \mu_j v_j \quad \text{and} \quad \lambda_{0k} := \sum_{j=1}^{m} \mu_j \lambda_{jk}, \qquad k = 1, \ldots, n.$$

Then $\lambda_{01} = \nu$, and the equation

$$\sum_{k=1}^{n} \lambda_{0k} u_k = v_0 \tag{5.4}$$

is a consequence of (5.1). For each $j = 1, \ldots, m$, apply λ_{j1} to both sides of (5.4) and subtract the resulting equation from (5.3). In this way we obtain

$$\sum_{k=2}^{n} (\lambda_{01}\lambda_{jk} - \lambda_{j1}\lambda_{0k}) u_k = \lambda_{01} v_j - \lambda_{j1} v_0, \qquad j = 1, \ldots, m. \tag{5.5}$$

The system consisting of the equations in (5.5) and the equation in (5.4) is equivalent to the original system of equations in (5.1).

Now let us show that (5.5) is uniquely solvable for $(u_2, \ldots, u_n)$. By the induction hypothesis, it suffices to verify that (5.5) satisfies the two conditions stated in the theorem. First, since the original system (5.1) is compatible, so is (5.5). Second, the homogeneous system associated to (5.5) only has the trivial solution. Indeed, if $(u_2, \ldots, u_n)$ is a nontrivial solution of the homogeneous system, then we can find $u_1 \in V$ such that

$$\lambda_{01} u_1 = -(\lambda_{02} u_2 + \cdots + \lambda_{0n} u_n),$$

because $\lambda_{01} = \nu$ is invertible. Thus, $(u_1, u_2, \ldots, u_n)$ would be a nontrivial solution to the homogeneous system associated with (5.1), which is a contradiction.

We have proved that (5.5) is uniquely solvable. Let $(u_2, \ldots, u_n)$ be the solution. Since $\lambda_{01} = \nu$ is invertible, we can find $u_1 \in V$ such that

$$\nu u_1 = v_0 - \sum_{k=2}^{n} \lambda_{0k} u_k.$$

Consequently, $(u_1, u_2, \ldots, u_n)$ is the unique solution of (5.1). ∎

Next, we discuss two special linear operator equations: linear partial difference equations and linear partial differential equations. Theorem 5.1 will be used to give criteria for solvability of those equations.

Let $\Pi(\mathbb{C}^s)$ denote the linear space of all polynomials of s variables with coefficients in $\mathbb{C}$. For a nonnegative integer d, we denote by $\Pi_d(\mathbb{C}^s)$ the subspace of all polynomials of (total) degree less than or equal to d. If no ambiguity arises, we write Π for $\Pi(\mathbb{C}^s)$ and Π_d for $\Pi_d(\mathbb{C}^s)$, respectively.

A mapping a from $\mathbb{Z}^s$ to $\mathbb{C}$ is called a **polynomial sequence**, if there is a polynomial q of s variables with coefficients in $\mathbb{C}$ such that $a(\alpha) = q(\alpha)$ for all

$\alpha \in \mathbb{Z}^s$. The **degree** of a is the same as the degree of q. Let $\mathbb{P}(\mathbb{Z}^s)$ denote the linear space of all polynomial sequences on $\mathbb{Z}^s$.

Suppose $p(z) = \sum_{\alpha\in\mathbb{Z}^s} c_\alpha z^\alpha$ is a Laurent polynomial of s variables with coefficients in $\mathbb{C}$, where $c_\alpha = 0$ except for finitely many α. Let e denote the s-tuple $(1,\ldots,1)$. Then $p(e) = \sum_{\alpha\in\mathbb{Z}^s} c_\alpha$. The polynomial p induces the difference operator $p(\tau) = \sum_{\alpha\in\mathbb{Z}^s} c_\alpha \tau^\alpha$. It is easily seen that $p(\tau)$ maps $\mathbb{P}(\mathbb{Z}^s)$ to itself. For a sequence a on $\mathbb{Z}^s$ we have

$$(\tau^\alpha - 1)a = a(\cdot + \alpha) - a.$$

Hence $(\tau^\alpha - 1)a = 0$ if a is a constant sequence. Moreover, if a is a polynomial sequence, then $(\tau^\alpha - 1)a$ is also a polynomial sequence of degree less than the degree of a. Thus, if $p(e) = 0$, then the difference operator $p(\tau)$ is degree-reducing; that is, for any polynomial sequence a, $p(\tau)a$ is a polynomial sequence of degree less than the degree of a. Consequently, $p(\tau)$ is invertible on $\mathbb{P}(\mathbb{Z}^s)$ if and only if $p(e) \neq 0$. Indeed, if $p(e) = 0$, then $p(\tau)a = 0$ for any constant sequence a. If $p(e) \neq 0$, then we can write $p = c - p_0$, where $c = p(e)$ and $p_0(e) = 0$. Thus, $p_0(\tau)$ is degree-reducing, and so $p_0^{n+1}(\tau)a = 0$ for all polynomial sequences a of degree n. Given a polynomial sequence a of degree n, the equation $p(\tau)r = a$ has a unique solution

$$r = \left[1/c + p_0(\tau)/c^2 + \cdots + p_0^n(\tau)/c^{n+1}\right]a.$$

This shows that $p(\tau)$ is invertible.

Let Λ be the ring of all partial difference operators of the form $p(\tau)$, where p is a Laurent polynomial of s variables with coefficients in $\mathbb{C}$. Then Λ is a commutative ring with identity. If I is a finitely generated ideal of Λ with $\ker I = \{0\}$, then I is invertible. To see this, let I be the ideal generated by $p_1(\tau),\ldots,p_m(\tau)$. If $\ker I = \{0\}$, then for at least one j, $p_j(e) \neq 0$, for otherwise the constant sequences would lie in the kernel of I. But $p_j(e) \neq 0$ implies that $p_j(\tau)$ is invertible. This shows that I is invertible.

THEOREM 5.2: *Let p_{jk} $(j = 1,\ldots,m; k = 1,\ldots,n)$ be Laurent polynomials of s variables with coefficients in $\mathbb{C}$. The homogeneous system of linear partial difference equations*

$$\sum_{k=1}^{n} p_{jk}(\tau)u_k = 0, \qquad j = 1,\ldots,m, \tag{5.6}$$

only has the trivial solution if and only if the matrix

$$P := \left(p_{jk}(e)\right)_{1\le j\le m, 1\le k\le n}$$

has rank n. Consequently, for given polynomial sequences $v_1, \dots, v_m$, the system of equations

$$\sum_{k=1}^{n} p_{jk}(\tau)u_k = v_j, \qquad j = 1, \dots, m,$$

is uniquely solvable for $(u_1, \dots, u_n) \in \mathbb{P}(\mathbb{Z}^s)^n$ if and only if the matrix P has rank n and the system is compatible.

Proof: If the rank of P is less than n, then there exists a nonzero vector $(a_1, \dots, a_n)$ in $\mathbb{C}^n \setminus \{0\}$ such that

$$\sum_{k=1}^{n} p_{jk}(e)a_k = 0 \qquad \text{for } j = 1, \dots, m. \tag{5.7}$$

For each k, let u_k be the constant sequence $\alpha \mapsto a_k$, $\alpha \in \mathbb{Z}^s$. Then $(u_1, \dots, u_n)$ is a nontrivial solution to the homogeneous system (5.6).

Conversely, suppose that the homogeneous system (5.6) has a nontrivial solution $(u_1, \dots, u_n)$. We observe that for any polynomial q, $(q(\tau)u_1, \dots, q(\tau)u_n)$ is also a solution of (5.6). We can find a polynomial q such that $q(\tau)u_1, \dots, q(\tau)u_n$ are constant sequences but $q(\tau)u_k \neq 0$ for at least one k. Let $a_k = q(\tau)u_k(0)$ for $k = 1, \dots, n$. Then the complex vector $(a_1, \dots, a_n)$ satisfies (5.7). Hence the rank of the matrix P is less than n. This proves the first part of the theorem.

The second part of the theorem follows immediately from the first part of the theorem and Theorem 5.1. ∎

The rest of this section is devoted to a study of linear partial differential equations. For this purpose we need the multi-index notation. Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An element in $\mathbb{N}_0^s$ is called a **multi-index**. If $\alpha = (\alpha_1, \dots, \alpha_s)$ is a multi-index, then its length $|\alpha|$ is defined by $|\alpha| := \alpha_1 + \cdots + \alpha_s$, and its factorial is defined by $\alpha! := \alpha_1! \cdots \alpha_s!$. For two multi-indices $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\beta = (\beta_1, \dots, \beta_s)$, by $\alpha \leq \beta$, or $\beta \geq \alpha$, we mean $\alpha_j \leq \beta_j$ for $j = 1, \dots, s$.

Let $\alpha \in \mathbb{N}_0^s$ be a multi-index. The differential operator D^α on $\Pi(\mathbb{C}^s)$ is defined by

$$D^\alpha \left(\sum_{\beta \in \mathbb{N}_0^s} b_\beta z^\beta \right) := \sum_{\beta \geq \alpha} b_\beta \frac{\beta!}{(\beta - \alpha)!} z^{\beta - \alpha}.$$

A polynomial $p = \sum_{\alpha \in \mathbb{N}_0^s} a_\alpha z^\alpha$ induces the differential operator $\sum_{\alpha \in \mathbb{N}_0^s} a_\alpha D^\alpha$, which is denoted by $p(D)$.

The differential operator $p(D)$ is invertible on Π if and only if $p(0) \neq 0$. Indeed, if $p(0) = 0$, then $p(D)1 = 0$. Conversely, if $p(0) \neq 0$, then we may write $p = c - p_0$, where $c = p(0)$ and p_0 is a polynomial with $p_0(0) = 0$. Then for any polynomial q of degree n, the equation $p(D)r = q$ has a unique solution

$$r = \left[1/c + p_0(D)/c^2 + \cdots + p_0^n(D)/c^{n+1}\right]q.$$

This shows that $p(D)$ is invertible.

Let Λ be the ring of all linear partial differential operators of the form $p(D)$, where p is a polynomial of s variables with coefficients in $\mathbb{C}$. Then Λ is a commutative ring with identity. If I is a finitely generated ideal of Λ with $\ker I = \{0\}$, then I is invertible. To see this, let I be the ideal generated by $p_1(D), \ldots, p_m(D)$. If $\ker I = \{0\}$, then $p_j(0) \neq 0$ for at least one j, for otherwise the constants would lie in $\ker I$. But $p_j(0) \neq 0$ implies that $p_j(D)$ is invertible on Π. This shows that I is invertible.

The following theorem can be proved in the same way as Theorem 5.2 was done.

THEOREM 5.3: *Let p_{jk} ($j = 1, \ldots, m; k = 1, \ldots, n$) be polynomials of s variables with coefficients in $\mathbb{C}$. The homogeneous system of linear partial differential equations*

$$\sum_{k=1}^{n} p_{jk}(D)u_k = 0, \qquad j = 1, \ldots, m,$$

only has the trivial solution if and only if the matrix

$$P := \left(p_{jk}(0)\right)_{1 \le j \le m, 1 \le k \le n}$$

has rank n. Consequently, for given polynomials $v_1, \ldots, v_m$, the system of equations

$$\sum_{k=1}^{n} p_{jk}(D)u_k = v_j, \qquad j = 1, \ldots, m,$$

is uniquely solvable for $(u_1, \ldots, u_n) \in \Pi^n$ if and only if the matrix P has rank n and the system is compatible.

6. Discrete convolution equations

In this section we shall give a criterion for solvability of discrete convolution equations.

Recall that $\ell(\mathbb{Z}^s)$ is the linear space of complex-valued sequences on $\mathbb{Z}^s$, and $\ell_0(\mathbb{Z}^s)$ is the linear space of all finitely supported sequences on $\mathbb{Z}^s$. Moreover, we use $c_0(\mathbb{Z}^s)$ to denote the linear space of all sequences a on $\mathbb{Z}^s$ such that

$\lim_{|\alpha|\to\infty} a(\alpha) = 0$, where $|\alpha| := |\alpha_1| + \cdots + |\alpha_s|$ for $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$. Given a sequence a on $\mathbb{Z}^s$, we define

$$\|a\|_p := \left(\sum_{\alpha\in\mathbb{Z}^s} |a(\alpha)|^p \right)^{1/p}, \qquad 1 \le p < \infty.$$

For $p = \infty$, we define $\|a\|_\infty$ to be the supremum of $\{|a(\alpha)|\colon \alpha \in \mathbb{Z}^s\}$. For $1 \le p \le \infty$ we denote by $\ell_p(\mathbb{Z}^s)$ the Banach space of all sequences a on $\mathbb{Z}^s$ such that $\|a\|_p < \infty$.

Given $a \in \ell(\mathbb{Z}^s)$, the formal Laurent series $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) z^\alpha$ is called the **symbol** of a, and denoted by $\tilde{a}(z)$. If $a \in \ell_1(\mathbb{Z}^s)$, then the symbol $\tilde{a}$ is a continuous function on the torus

$$\mathbb{T}^s := \{(z_1, \ldots, z_s) \in \mathbb{C}^s\colon |z_1| = \cdots = |z_s| = 1\}.$$

If $a \in \ell_0(\mathbb{Z}^s)$, then $\tilde{a}$ is a Laurent polynomial.

For $a, b \in \ell(\mathbb{Z}^s)$, we define the **convolution** of a and b by

$$a*b(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha-\beta) b(\beta), \qquad \alpha \in \mathbb{Z}^s,$$

whenever the above series is absolutely convergent. For example, if δ is the sequence given by $\delta(\alpha) = 1$ for $\alpha = 0$ and $\delta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \setminus \{0\}$, then $a*\delta = a$ for all $a \in \ell(\mathbb{Z}^s)$. Evidently, for $a \in \ell_0(\mathbb{Z}^s)$ and $b \in \ell(\mathbb{Z}^s)$, the convolution $a*b$ is well defined.

Let a be an element in $\ell_0(\mathbb{Z}^s)$ such that $\tilde{a}(z) \ne 0$ for all $z \in \mathbb{T}^s$. For given $v \in \ell_\infty(\mathbb{Z}^s)$, the discrete convolution equation

$$a*u = v$$

has a unique solution for $u \in \ell_\infty(\mathbb{Z}^s)$. To see this, let

$$c(\alpha) := \frac{1}{(2\pi)^s} \int_{[0,2\pi)^s} \frac{1}{\tilde{a}(e^{i\xi})} e^{-i\alpha\cdot\xi} d\xi, \qquad \alpha \in \mathbb{Z}^s.$$

Then the sequence c decays exponentially fast, and $\tilde{c}(z)\tilde{a}(z) = 1$ for all $z \in \mathbb{T}^s$. Hence $c*a = \delta$. If $a*u = v$, then it follows that

$$u = \delta*u = (c*a)*u = c*(a*u) = c*v.$$

This proves uniqueness of the solution. Moreover, if v lies in $\ell_p(\mathbb{Z}^s)$ for some p, $1 \le p \le \infty$, then the solution u lies in $\ell_p(\mathbb{Z}^s)$; if $v \in c_0(\mathbb{Z}^s)$, then the solution u also lies in $c_0(\mathbb{Z}^s)$.

Consider the system of discrete convolution equations

$$(6.1)\qquad \sum_{k=1}^{n} a_{jk}*u_k = v_j, \qquad j=1,\ldots,m,$$

where $a_{jk} \in \ell_0(\mathbb{Z}^s)$ $(j=1,\ldots,m; k=1,\ldots,n)$ and $v_j \in \ell(\mathbb{Z}^s)$ $(j=1,\ldots,m)$. We say that this system of equations is **compatible** if for any $c_1,\ldots,c_m \in \ell_0(\mathbb{Z}^s)$ with $\sum_{j=1}^m c_j*a_{jk}=0$, $k=1,\ldots,n$, one must have $\sum_{j=1}^m c_j*v_j=0$.

THEOREM 6.1: *Let $v_1,\ldots,v_m \in \ell_\infty(\mathbb{Z}^s)$. Suppose that the system of discrete convolution equations in (6.1) is compatible. If the matrix*

$$A(z) := \left(\tilde{a}_{jk}(z)\right)_{1\le j\le m, 1\le k\le n}$$

has rank n for every $z \in \mathbb{T}^s$, then the system of equations in (6.1) is uniquely solvable for $(u_1,\ldots,u_n) \in (\ell_\infty(\mathbb{Z}^s))^n$. Furthermore, if $v_1,\ldots,v_m$ lie in $\ell_p(\mathbb{Z}^s)$ for some p, $1\le p<\infty$, then the solutions $u_1,\ldots,u_n$ also lie in $\ell_p(\mathbb{Z}^s)$; if $v_1,\ldots,v_m$ lie in $c_0(\mathbb{Z}^s)$, then the solutions $u_1,\ldots,u_n$ also lie in $c_0(\mathbb{Z}^s)$.

Proof: For $j=1,\ldots,m$, let c_j be the sequence given by $c_j(\alpha)=\overline{a_{j1}(-\alpha)}$, $\alpha\in\mathbb{Z}^s$. Then $\tilde{c}_j(z)=\overline{\tilde{a}_{j1}(z)}$ for $z\in\mathbb{T}^s$. Let $a_{0k} := \sum_{j=1}^m c_j*a_{jk}$, $k=1,\ldots,n$. Since $A(z)$ has rank n, the Laurent polynomials $\tilde{a}_{11}(z),\ldots,\tilde{a}_{m1}(z)$ do not have common zeros in $\mathbb{T}^s$. Hence

$$\tilde{a}_{01}(z) = \sum_{j=1}^m \tilde{c}_j(z)\tilde{a}_{j1}(z) = \sum_{j=1}^m \left|\tilde{a}_{j1}(z)\right|^2 > 0 \qquad \forall z\in\mathbb{T}^s.$$

Let us consider the case $n=1$ first. In this case, (6.1) implies

$$a_{01}*u_1 = \sum_{j=1}^m c_j*v_j =: v_0.$$

Since $\tilde{a}_{01}(z)>0$ for all $z\in\mathbb{T}^s$, the equation $a_{01}*u_1=v_0$ is uniquely solvable for u in $\ell_\infty(\mathbb{Z}^s)$. Let u_1 be the solution. By the assumption, the original system of equations in (6.1) is compatible; hence $a_{01}*v_j=a_{j1}*v_0$. It follows that $a_{01}*a_{j1}*u_1 = a_{j1}*v_0 = a_{01}*v_j$. Therefore, $a_{j1}*u_1=v_j$ for $j=1,\ldots,m$. This shows that u_1 is the unique solution to the system of equations in (6.1).

The proof proceeds with induction on n. Suppose $n>1$ and the desired result is valid for $n-1$. Let c_j $(j=1,\ldots,m)$ and a_{0k} $(k=1,\ldots,n)$ be the same sequences as in the above. Let $w_0 := v_0 = \sum_{j=1}^m c_j*v_j$, $w_j := a_{01}*v_j - a_{j1}*w_0$ $(j=1,\ldots,m)$, and

$$b_{jk} := a_{01}*a_{jk} - a_{j1}*a_{0k}, \qquad j=1,\ldots,m; k=2,\ldots,n.$$

Then $w_0, w_1, \ldots, w_m \in \ell_\infty(\mathbb{Z}^s)$. Consequently, (6.1) is equivalent to the following system of equations:

$$\sum_{k=1}^{n} a_{0k} * u_k = w_0 \tag{6.2}$$

and

$$\sum_{k=2}^{n} b_{jk} * u_k = w_j, \qquad j = 1, \ldots, m. \tag{6.3}$$

We observe that (6.3) is compatible and the matrix $B(z) := (\tilde{b}_{jk}(z))_{1 \le j \le m, 2 \le k \le n}$ has rank $n-1$ for every $z \in \mathbb{T}^s$. Thus, by the induction hypothesis, (6.3) is uniquely solvable for $u_2, \ldots, u_n$ in $\ell_\infty(\mathbb{Z}^s)$. Once $u_2, \ldots, u_n$ are obtained, u_1 is uniquely determined from (6.2). This completes the induction procedure.

Finally, if $v_1, \ldots, v_m$ lie in $\ell_p(\mathbb{Z}^s)$ for some p, $1 \le p < \infty$, then the above proof shows that the solutions $u_1, \ldots, u_n$ also lie in $\ell_p(\mathbb{Z}^s)$. The same conclusion holds true for $c_0(\mathbb{Z}^s)$. ∎

7. Stable generators

Let Φ be a finite subset of $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. In this section, we shall characterize $S_p(\Phi)$ in terms of the semi-convolutions of the generators with sequences in $\ell_p(\mathbb{Z}^s)$, if the shifts of the functions in Φ are stable. If, in addition, the functions in Φ are compactly supported, we shall prove $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R}^s)$ for $1 \le p < \infty$. When $p = \infty$, we denote by $L_{\infty,0}(\mathbb{R}^s)$ the subspace of $L_\infty(\mathbb{R}^s)$ consisting of all functions $f \in L_\infty(\mathbb{R}^s)$ such that $\|f\|_\infty(\mathbb{R}^s \setminus [-k,k]^s) \to 0$ as $k \to \infty$. We shall prove $S_\infty(\Phi) = S(\Phi) \cap L_{\infty,0}(\mathbb{R}^s)$.

Let Φ be a finite subset of $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. We say that the shifts $\phi(\cdot - \alpha)$ $(\phi \in \Phi, \alpha \in \mathbb{Z}^s)$ are L_p-**stable** if there are two positive constants C_1 and C_2 such that

$$C_1 \sum_{\phi \in \Phi} \|a_\phi\|_p \le \Big\| \sum_{\phi \in \Phi} \phi *' a_\phi \Big\|_p \le C_2 \sum_{\phi \in \Phi} \|a_\phi\|_p$$

for all sequences $a_\phi \in \ell_0(\mathbb{Z}^s)$, $\phi \in \Phi$. Under some mild decay conditions on the functions in Φ, it was proved by Jia and Micchelli ([13] and [14]) that the shifts of the functions in Φ are L_p-stable if and only if for any $\xi \in \mathbb{R}^s$, the sequences $(\hat{\phi}(\xi + 2\pi\beta))_{\beta \in \mathbb{Z}^s}$ $(\phi \in \Phi)$ are linearly independent. When $p = 2$, their results were generalized by de Boor, DeVore and Ron in [4].

Suppose $\Phi = \{\phi_1, \ldots, \phi_n\}$. Let T_Φ be the mapping from $(\ell_0(\mathbb{Z}^s))^n$ to $L_p(\mathbb{R}^s)$ given by

$$T_\Phi(a_1, \ldots, a_n) := \sum_{k=1}^{n} \phi_k *' a_k, \qquad a_1, \ldots, a_n \in \ell_0(\mathbb{Z}^s).$$

Let $X := (\ell_p(\mathbb{Z}^s))^n$ for $1 \le p < \infty$ and $X := (c_0(\mathbb{Z}^s))^n$ for $p = \infty$. The norm on X is defined by

$$\|(a_1, \ldots, a_n)\|_X := \sum_{k=1}^{n} \|a_k\|_p.$$

Suppose that the shifts of the functions in Φ are stable. Then the domain of T_Φ can be extended to X, and T_Φ is a one-to-one continuous linear operator from X to $Y := L_p(\mathbb{R}^s)$. For $a = (a_1, \ldots, a_n) \in X$, we write $\sum_{k=1}^{n} \phi_k *' a_k$ for $T_\Phi(a)$. In other words,

$$\lim_{N \to \infty} \left\| \sum_{k=1}^{n} \phi_k *' a_k - \sum_{k=1}^{n} \sum_{|\alpha| \le N} \phi_k(\cdot - \alpha) a_k(\alpha) \right\|_p = 0.$$

Moreover, there exists a positive constant C such that $C\|a\|_X \le \|T_\Phi(a)\|_Y$ for all $a \in X$. From a well-known result in functional analysis (see, e.g., [18, p. 70]), the range of T_Φ is closed. In other words, $T_\Phi(X) = S_p(\Phi)$. Thus, we have the following result.

THEOREM 7.1: *Let Φ be a finite subset of $L_p(\mathbb{R}^s)$ such that the shifts of the functions in Φ are L_p-stable ($1 \le p \le \infty$). For $1 \le p < \infty$, a function f lies in $S_p(\Phi)$ if and only if*

$$f = \sum_{\phi \in \Phi} \phi *' a_\phi$$

for some sequences a_ϕ in $\ell_p(\mathbb{Z}^s)$. For $p = \infty$, a function f lies in $S_\infty(\Phi)$ if and only if $f = \sum_{\phi \in \Phi} \phi *' a_\phi$ for some sequences a_ϕ in $c_0(\mathbb{Z}^s)$.

Theorem 7.1 does not apply to the case in which the stability condition is not satisfied. For example, let $\phi := \chi - \chi(\cdot - 1)$, where χ is the characteristic function of $[0, 1)$. Then $\chi \in S_2(\phi)$ (see [4, Example 2.7]), but χ cannot be written in the form $\chi = \phi *' a$ for any $a \in \ell_2(\mathbb{Z})$. Indeed, if a is an element of $\ell_2(\mathbb{Z})$, then the 2π-periodic function $\xi \mapsto \tilde{a}(e^{i\xi})$ is square integrable on $[0, 2\pi)$ and

$$\widehat{\phi *' a}(\xi) = \hat{\phi}(\xi)\tilde{a}(e^{-i\xi}) = \hat{\chi}(\xi)(1 - e^{-i\xi})\tilde{a}(e^{-i\xi}).$$

Thus, $\chi = \phi *' a$ implies that

$$\tilde{a}(e^{i\xi}) = 1 \big/ (1 - e^{i\xi}) \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

But the function $\xi \mapsto 1/(1-e^{i\xi})$ is not square integrable on $[0,2\pi)$. This contradiction verifies our claim. Moreover, we have $\int_{\mathbb{R}} \chi(x)dx = 1$ and $\chi = \sum_{j=0}^{\infty} \phi(\cdot - j) \in S(\phi)$. However, any function f in $S_0(\phi)$ satisfies $\int_{\mathbb{R}} f(x)dx = 0$. Since $S_1(\phi)$ is the closure of $S_0(\phi)$ in $L_1(\mathbb{R})$, we also have $\int_{\mathbb{R}} f(x)dx = 0$ for all $f \in S_1(\phi)$. This shows that $\chi \notin S_1(\phi)$. Therefore $S_1(\phi) \neq S(\phi) \cap L_1(\mathbb{R})$.

When Φ is a finite collection of compactly supported functions in $L_p(\mathbb{R})$, it was shown in [11] that $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R})$ for $1 < p < \infty$ and $S_\infty(\Phi) = S(\Phi) \cap L_{\infty,0}(\mathbb{R})$. The following theorem gives a similar result for $s > 1$ if the shifts of the functions in Φ are stable.

THEOREM 7.2: *Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. If the shifts of the functions in Φ are L_p-stable, then $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R}^s)$ for $1 \le p < \infty$, and $S_\infty(\Phi) = S(\Phi) \cap L_{\infty,0}(\mathbb{R}^s)$.*

Proof: By Theorem 3.1, $S(\Phi) \cap L_p(\mathbb{R}^s)$ is closed in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. Hence $S_p(\Phi)$ is contained in $S(\Phi) \cap L_p(\mathbb{R}^s)$. For $p = \infty$, we also have $S_\infty(\Phi) \subseteq S(\Phi) \cap L_{\infty,0}(\mathbb{R}^s)$.

Suppose $\Phi = \{\phi_1, \ldots, \phi_n\}$. We can find functions $\psi_1, \ldots, \psi_m \in L_p(\mathbb{R}^s)$ such that they vanish outside the unit cube $[0,1)^s$ and $\{\psi_j|_{[0,1)^s} : j = 1, \ldots, m\}$ forms a basis for $S(\Phi)|_{[0,1)^s}$. Then each ϕ_k $(k = 1, \ldots, n)$ can be represented as

$$\phi_k = \sum_{j=1}^{m} \psi_j *' a_{jk}, \tag{7.1}$$

where a_{jk} $(j = 1, \ldots, m; k = 1, \ldots, n)$ are finitely supported sequences on $\mathbb{Z}^s$.

A function $f \in S(\Psi)$ has the following representation:

$$f = \sum_{j=1}^{m} \psi_j *' v_j, \tag{7.2}$$

where $v_1, \ldots, v_m$ are sequences on $\mathbb{Z}^s$. If f lies in $L_p(\mathbb{R}^s)$ for $1 \le p < \infty$, then the sequences $v_1, \ldots, v_m$ lie in $\ell_p(\mathbb{Z}^s)$. To see this, we observe that, for $\beta \in \mathbb{Z}^s$,

$$f(x) = \sum_{j=1}^{m} v_j(\beta)\psi_j(x - \beta) \qquad \text{for } x \in \beta + [0,1)^s.$$

Hence, there exists a constant $C > 0$ such that

$$|v_j(\beta)|^p \le C^p \int_{\beta+[0,1)^s} |f(x)|^p dx \qquad \forall j = 1, \ldots, m \text{ and } \beta \in \mathbb{Z}^s.$$

It follows that $\|v_j\|_p \le C\|f\|_p$ for $j = 1, \ldots, m$. Thus, $v_1, \ldots, v_m$ lie in $\ell_p(\mathbb{Z}^s)$. Similarly, if $f \in L_{\infty,0}(\mathbb{R}^s)$, then $v_1, \ldots, v_m$ lie in $c_0(\mathbb{Z}^s)$.

Now assume that $f \in S(\Phi)$. Then there exist sequences $u_1, \ldots, u_n$ on $\mathbb{Z}^s$ such that $f = \sum_{k=1}^n \phi_k *' u_k$. This in connection with (7.1) and (7.2) tells us that $u_1, \ldots, u_n$ satisfy the following system of discrete convolution equations:

$$\sum_{k=1}^{n} a_{jk} * u_k = v_j, \qquad j = 1, \ldots, m. \tag{7.3}$$

Consequently, this system of equations is compatible. We shall show that the matrix

$$A(z) := \left(\tilde{a}_{jk}(z)\right)_{1\le j\le m, 1\le k\le n}$$

has rank n for every $z \in \mathbb{T}^s$, provided that the shifts of $\phi_1, \ldots, \phi_n$ are stable. For this purpose, we deduce from (7.1) that for $k = 1, \ldots, n$,

$$\hat{\phi}_k(\xi + 2\pi\beta) = \sum_{j=1}^{m} \tilde{a}_{jk}(e^{-i\xi})\hat{\psi}_j(\xi + 2\pi\beta), \qquad \xi \in \mathbb{R}^s, \beta \in \mathbb{Z}^s.$$

If $A(e^{-i\xi})$ had rank less than n for some $\xi \in \mathbb{R}^s$, then the sequences $(\hat{\phi}_k(\xi + 2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $k = 1, \ldots, n$, would be linearly dependent, which contradicts the assumption on stability. Since $A(z)$ has rank n for every $z \in \mathbb{T}^s$ and the system of equations in (7.3) is compatible, we conclude that (7.3) is uniquely solvable for $u_1, \ldots, u_n$ in $\ell_p(\mathbb{Z}^s)$, by Theorem 6.1. Let $(u_1, \ldots, u_n)$ be the solution. Then $f = \sum_{k=1}^n \phi_k *' u_k$ lies in $S_p(\Phi)$. This shows that $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R}^s)$ for $1 \le p < \infty$.

If $f \in S(\Phi) \cap L_{\infty,0}(\mathbb{R}^s)$, then the sequences $v_1, \ldots, v_m$ lie in $c_0(\mathbb{Z}^s)$; hence $u_1, \ldots, u_n$ lie in $c_0(\mathbb{Z}^s)$. This shows that $S_\infty(\Phi) = S(\Phi) \cap L_{\infty,0}(\mathbb{R}^s)$. ∎

8. Approximation order

In this section we shall apply the results on linear operator equations to a study of approximation by shift-invariant spaces. See [10] for a recent survey on this topic.

For a subset E of $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ and $f \in L_p(\mathbb{R}^s)$, define the distance from f to E by

$$\operatorname{dist}(f, E)_p := \inf_{g\in E}\{\|f - g\|_p\}.$$

Let S be a closed shift-invariant subspace of $L_p(\mathbb{R}^s)$. For $h > 0$, let σ_h be the scaling operator given by the equation $\sigma_h f := f(\cdot/h)$ for functions f on

$\mathbb{R}^s$. Let $S^h := \sigma_h(S)$. For a real number $r > 0$, we say that S provides L_p-**approximation order** r if, for every sufficiently smooth function f in $L_p(\mathbb{R}^s)$,

$$\operatorname{dist}(f, S^h)_p \le C_f h^r \qquad \forall h > 0,$$

where C_f is a constant independent of h. We say that S provides L_p-**density order** r if

$$\lim_{h\to 0^+} \operatorname{dist}(f, S^h)_p / h^r = 0.$$

Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$). We say that $S(\Phi)$ provides approximation order r (resp. density order r) if $S(\Phi) \cap L_p(\mathbb{R}^s)$ does.

Let r be a positive integer, and let ϕ be a compactly supported function in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$) with $\hat{\phi}(0) \ne 0$. Then $S(\phi)$ provides approximation order r if and only if $S(\phi)$ contains Π_{r-1}. This result was established by Ron [17] for the case $p = \infty$, and by Jia [9] for the general case $1 \le p \le \infty$. The following theorem extends their results to finitely generated shift-invariant spaces.

THEOREM 8.1: *Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ ($1 \le p \le \infty$). Suppose that the sequences $(\hat{\phi}_k(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $k = 1, \ldots, n$, are linearly independent. For a positive integer r, the following statements are equivalent:*

(a) *$S(\Phi)$ provides L_p-approximation order r.*
(b) *$S(\Phi)$ provides L_p-density order $r - 1$.*
(c) *$S(\Phi) \supseteq \Pi_{r-1}$.*
(d) *There exists a function $\psi \in S_0(\Phi)$ such that*

$$\sum_{\alpha\in\mathbb{Z}^s} q(\alpha)\psi(\cdot - \alpha) = q \qquad \forall q \in \Pi_{r-1}. \tag{8.1}$$

It is known that (8.1) is true if and only if for all $\nu \in \mathbb{N}_0^s$ with $|\nu| < r$ and all $\beta \in \mathbb{Z}^s$

$$D^\nu \hat{\psi}(2\pi\beta) = \delta_{0\nu}\delta_{0\beta},$$

where δ stands for the Kronecker sign. This result was first established by Schoenberg [19] for the univariate case, and then extended by Strang and Fix [20] to the multivariate case.

If the shifts of the functions in Φ are stable, then, for each $\xi \in \mathbb{R}^s$, the sequences $(\hat{\phi}_k(\xi + 2\pi\beta))_{\beta\in\mathbb{Z}^s}$ ($k = 1, \ldots, n$) are linearly independent. Thus, the conclusion of Theorem 8.1 is valid if the shifts of the functions in Φ are stable. This weaker form of Theorem 8.1 was first established by Lei, Jia and Cheney [16].

Suppose Φ is contained in $L_2(\mathbb{R}^s)$. Recall that the bracket product $[\phi_j, \phi_k]$ is given by

$$[\phi_j, \phi_k](e^{i\xi}) = \sum_{\beta \in \mathbb{Z}^s} \hat{\phi}_j(\xi + 2\pi\beta)\overline{\hat{\phi}_k(\xi + 2\pi\beta)}, \qquad \xi \in \mathbb{R}^s.$$

Define the Gram matrix G_Φ by

$$G_\Phi(\xi) := \left([\phi_j, \phi_k](e^{i\xi})\right)_{1 \le j,k \le n}, \qquad \xi \in \mathbb{R}^s.$$

Then the sequences $(\hat{\phi}_k(2\pi\beta))_{\beta \in \mathbb{Z}^s}$, $k = 1, \ldots, n$, are linearly independent if and only if $\det G_\Phi(0) \neq 0$.

In order to prove Theorem 8.1 we observe that (a) implies (b) trivially. It was proved in [9] that (b) implies (c). The implication (d) $\Rightarrow$ (a) is well known. See [12] for an explicit L_p-approximation scheme. It remains to prove (c) $\Rightarrow$ (d). This was proved by de Boor [2] for the case where Φ consists of a single function. For the general case, we need some auxiliary results about polynomials and polynomial sequences. Let T_Φ be the mapping given by

$$T_\Phi(q_1, \ldots, q_n) := \sum_{k=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} \phi_k(\cdot - \alpha) q_k(\alpha), \qquad \text{for } (q_1, \ldots, q_n) \in \Pi^n.$$

LEMMA 8.2: *Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a collection of integrable functions on $\mathbb{R}^s$ with compact support. Then the following conditions are equivalent.*

(a) *The sequences $(\hat{\phi}_k(2\pi\beta))_{\beta \in \mathbb{Z}^s}$, $k = 1, \ldots, n$, are linearly independent.*
(b) *$T_\Phi(q_1, \ldots, q_n) = 0$ for polynomials $q_1, \ldots, q_n$ implies $q_1 = \cdots = q_n = 0$.*
(c) *Any polynomial $q \in S(\Phi)$ can be uniquely represented as $T_\Phi(q_1, \ldots, q_n)$ for some polynomials $q_1, \ldots, q_n$.*

Proof: As in the proof of Theorem 7.2, there exist functions $\psi_1, \ldots, \psi_m \in L_1(\mathbb{R}^s)$ such that they vanish outside the unit cube $[0,1)^s$ and $\{\psi_j|_{[0,1)^s} : j = 1, \ldots, m\}$ forms a basis for $S(\Phi)|_{[0,1)^s}$. Then each ϕ_k ($k = 1, \ldots, n$) can be represented as

$$\phi_k = \sum_{j=1}^{m} \psi_j *' a_{jk}, \tag{8.2}$$

where a_{jk} ($j = 1, \ldots, m; k = 1, \ldots, n$) are finitely supported sequences on $\mathbb{Z}^s$.

Let

$$g_{jk}(z) := \sum_{\beta \in \mathbb{Z}^s} a_{jk}(\beta) z^{-\beta}, \qquad z \in (\mathbb{C} \setminus \{0\})^s.$$

For given $v_1, \ldots, v_m \in \ell(\mathbb{Z}^s)$, the function $f := \sum_{j=1}^m \psi_j *' v_j$ lies in $S(\Phi)$ if and only if the following system of linear partial difference equations

$$\sum_{k=1}^{n} g_{jk}(\tau) u_k = v_j, \qquad j = 1, \ldots, m, \tag{8.3}$$

is solvable for $(u_1, \ldots, u_n) \in (\ell(\mathbb{Z}^s))^n$.

Now we restrict the difference operators $g_{jk}(\tau)$ to the space $\mathbb{P}(\mathbb{Z}^s)$. From (8.2) we deduce that

$$\hat{\phi}_k(2\pi\beta) = \sum_{j=1}^{m} g_{jk}(e) \hat{\psi}_j(2\pi\beta), \qquad k = 1, \ldots, n,$$

where e is the s-tuple $(1, \ldots, 1)$. Since the shifts of $\psi_1, \ldots, \psi_m$ are linearly independent, the sequences $(\hat{\psi}_j(2\pi\beta))_{\beta \in \mathbb{Z}^s}$, $j = 1, \ldots, m$, are linearly independent (see [13]). Thus, the sequences $(\hat{\phi}_k(2\pi\beta))_{\beta \in \mathbb{Z}^s}$ $(k = 1, \ldots, n)$ are linearly independent if and only if the matrix $G := (g_{jk}(e))_{1 \le j \le m, 1 \le k \le n}$ has rank n. We observe that $T_\Phi(q_1, \ldots, q_n) = 0$ if and only if

$$\sum_{k=1}^{n} g_{jk}(\tau) q_k = 0.$$

By Theorem 5.2, we conclude that conditions (a) and (b) are equivalent.

Obviously, (c) implies (b). It remains to prove (b) implies (c). To this end, let $e_1, \ldots, e_s$ be the unit coordinate vectors in $\mathbb{R}^s$, and let ∇_t $(t = 1, \ldots, s)$ be the difference operator given by $\nabla_t f = f - f(\cdot - e_t)$. Let $q \in S(\Phi) \cap \Pi$ and assume that $q = \sum_{j=1}^m \psi_j *' v_j$ for some sequences $v_1, \ldots, v_m$. We claim that $v_1, \ldots, v_m$ are polynomial sequences. Indeed, if q is a polynomial of degree less than r, then

$$\sum_{j=1}^{m} \psi_j *' (\nabla_t^r v_j) = \nabla_t^r q = 0, \qquad t = 1, \ldots, s.$$

Since the shifts of $\psi_1, \ldots, \psi_m$ are linearly independent, we have $\nabla_t^r v_j = 0$ for $t = 1, \ldots, s$ and $j = 1, \ldots, m$. This shows that $v_1, \ldots, v_m$ are polynomial sequences. Since q lies in $S(\Phi)$, there exist sequences $u_1, \ldots, u_n$ satisfying the system (8.3) of linear partial difference equations; hence (8.3) is compatible. Moreover, condition (b) tells us that the associated homogeneous system of (8.3) only has the trivial solution. Thus, by Theorem 5.1, the system (8.3) is uniquely solvable for $(u_1, \ldots, u_n) \in \mathbb{P}(\mathbb{Z}^s)^n$. This shows that q can be uniquely represented as $T_\Phi(q_1, \ldots, q_n)$ for some polynomials $q_1, \ldots, q_n$. ∎

LEMMA 8.3: *Let F be a linear mapping from Π_r to Π. Suppose F commutes with the shift operators, that is,*

$$F\big(q(\cdot-\alpha)\big)=(Fq)(\cdot-\alpha)\qquad \forall q\in\Pi_r \text{ and } \alpha\in\mathbb{Z}^s.$$

Then there exists a polynomial $f\in\Pi_r$ such that

$$F(q)=f(\tau)q\qquad \forall q\in\Pi_r.$$

Proof: We use Δ_r to denote the set $\{\alpha\in\mathbb{N}_0^s\colon |\alpha|\le r\}$. For $\beta\in\mathbb{N}_0^s$, let q_β denote the monomial given by $q_\beta(z)=z^\beta$. We wish to find a polynomial $f\in\Pi_r$ such that

$$f(\tau)q_\beta(0)=c_\beta:=Fq_\beta(0)\qquad \forall\beta\in\Delta_r. \tag{8.4}$$

Suppose $f(z)=\sum_{\alpha\in\Delta_r}a_\alpha z^\alpha$. Then the above equation is equivalent to the following:

$$\sum_{\alpha\in\Delta_r}a_\alpha\alpha^\beta=c_\beta,\qquad \beta\in\Delta_r. \tag{8.5}$$

The matrix $(\alpha^\beta)_{\alpha,\beta\in\Delta_r}$ is nonsingular. Indeed, if b_β ($\beta\in\mathbb{Z}^s$) are complex numbers such that $\sum_{\beta\in\Delta_r}b_\beta\alpha^\beta=0$ for all $\alpha\in\Delta_r$, then $b_\beta=0$ for all $\beta\in\Delta_r$ (see, e.g., [1, §4]). Thus, there exists a unique vector $(a_\alpha)_{\alpha\in\Delta_r}$ satisfying (8.5). With a_α chosen in this way, the polynomial $f(z)=\sum_{\alpha\in\Delta_r}a_\alpha z^\alpha$ satisfies (8.4). Since the monomials q_β ($\beta\in\Delta_r$) span Π_r, it follows that $Fq(0)=f(\tau)q(0)$ for all $q\in\Pi_r$. For any $\gamma\in\mathbb{Z}^s$, we have

$$Fq(\gamma)=F(q(\cdot+\gamma))(0)=f(\tau)q(\cdot+\gamma)(0)=f(\tau)q(\gamma).$$

Thus, the two polynomials Fq and $f(\tau)q$ agree on $\mathbb{Z}^s$. Hence $Fq=f(\tau)q$ for all $q\in\Pi_r$. This completes the proof. ∎

Proof of Theorem 8.1: It remains to prove (c) ⇒ (d). Suppose $q\in\Pi_{r-1}$. Then $q\in S(\Phi)$, and by Lemma 8.2 there exist unique polynomials $q_1,\dots,q_n$ such that

$$q=\sum_{\alpha\in\mathbb{Z}^s}\sum_{k=1}^n\phi_k(\cdot-\alpha)q_k(\alpha).$$

For each k, the mapping $F_k\colon q\mapsto q_k$ is a linear mapping from Π_{r-1} to Π which commutes with shift operators. By Lemma 8.3 we can find a polynomial $f_k\in\Pi_{r-1}$ such that $F_kq=f_k(\tau)q$ for all $q\in\Pi_{r-1}$. It follows that, for each $q\in\Pi_{r-1}$,

$$\begin{aligned}q=\sum_{\alpha\in\mathbb{Z}^s}\sum_{k=1}^n\phi_k(\cdot-\alpha)f_k(\tau)q(\alpha)&=\sum_{\alpha\in\mathbb{Z}^s}\sum_{k=1}^n\big(f_k(\tau)\phi_k\big)(\cdot-\alpha)q(\alpha)\\&=\sum_{\alpha\in\mathbb{Z}^s}\psi(\cdot-\alpha)q(\alpha),\end{aligned}$$

where $\psi := \sum_{k=1}^{n} f_k(\tau)\phi_k$ belongs to $S_0(\Phi)$. This shows that (c) implies (d). ∎

References

[1] C. de Boor, *Topics in multivariate approximation theory*, in *Topics in Numerical Analysis* (P. Turner, ed.), Lecture Notes in Mathematics **965**, Springer-Verlag, Berlin, 1982, pp. 39–78.

[2] C. de Boor, *The polynomials in the linear span of integer translates of a compactly supported function*, Constructive Approximation **3** (1987), 199–208.

[3] C. de Boor, R. DeVore and A. Ron, *Approximation from shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Transactions of the American Mathematical Society **341** (1994), 787–806.

[4] C. de Boor, R. DeVore and A. Ron, *The structure of finitely generated shift-invariant subspaces in* $L_2(\mathbb{R}^d)$, Journal of Functional Analysis **119** (1994), 37–78.

[5] W. Dahmen and C. A. Micchelli, *Translates of multivariate splines*, Linear Algebra and its Applications **52** (1983), 217–234.

[6] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Tracts in Mathematics **17**, Wiley-Interscience, New York, 1970.

[7] G. B. Folland, *Real Analysis*, Wiley, New York, 1984.

[8] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[9] R. Q. Jia, *The Toeplitz theorem and its applications to Approximation Theory and linear PDE's*, Transactions of the American Mathematical Society **347** (1995), 2585–2594.

[10] R. Q. Jia, *Refinable shift-invariant spaces: from splines to wavelets*, in *Approximation Theory VIII*, Vol. 2 (C. K. Chui and L. L. Schumaker, eds.), Word Scientific Publishing Co., Inc., 1995, pp. 179–208.

[11] R. Q. Jia, *Shift-invariant spaces on the real line*, Proceedings of the American Mathematical Society **125** (1997), 785–793.

[12] R. Q. Jia and J. J. Lei, *Approximation by multiinteger translates of functions having global support*, Journal of Approximation Theory **72** (1993), 2–23.

[13] R. Q. Jia and C. A. Micchelli, *On linear independence of integer translates of a finite number of functions*, Proceedings of the Edinburgh Mathematical Society **36** (1992), 69-85.

[14] R. Q. Jia and C. A. Micchelli, *Using the refinement equation for the construction of pre-wavelets II: Powers of two*, in *Curves and Surfaces* (P. J. Laurent, A. Le Méhauté and L. L. Schumaker, eds.), Academic Press, New York, 1991, pp. 209–246.

[15] R. Q. Jia, S. Riemenschneider and Z. W. Shen, *Solvability of systems of linear operator equations*, Proceedings of the American Mathematical Society **120** (1994), 815–824.

[16] J. J. Lei, R. Q. Jia, and E. W. Cheney, *Approximation from shift-invariant spaces by integral operators*, SIAM Journal of Mathematical Analysis **28** (1997), 481–498.

[17] A. Ron, *A characterization of the approximation order of multivariate spline spaces*, Studia Mathematica **98** (1991), 73–90.

[18] M. Schechter, *Principles of Functional Analysis*, Academic Press, New York, 1971.

[19] I. J. Schoenberg, *Contributions to the problem of approximation of equidistant data by analytic functions*, Quarterly of Applied Mathematics **4** (1946), 45–99, 112–141.

[20] G. Strang and G. Fix, *A Fourier analysis of the finite-element variational method*, in *Constructive Aspects of Functional Analysis* (G. Geymonat, ed.), C.I.M.E., 1973, pp. 793–840.